

Classical theta constants vs. lattice theta series, and super string partition functions

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ABSTRACT: Recently, various possible expressions for the vacuum-to-vacuum superstring amplitudes has been proposed at genus $g = 3, 4, 5$. To compare the different proposals, here we will present a careful analysis of the comparison between the two main technical tools adopted to realize the proposals: the classical theta constants and the lattice theta series. We compute the relevant Fourier coefficients in order to relate the two spaces. We will prove the equivalence up to genus 4. In genus five we will show that the solutions are equivalent modulo the Schottky form and coincide if we impose the vanishing of the cosmological constant.

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1. Introduction

In the perturbative approach, superstring theory can be formulated using the path integral formalism outlined by Polyakov. The computation of amplitudes from first principles is an old problem in string theory and finds its roots in the correct mathematical definition of the theory. In a series of papers D'Hoker and Phong have obtained the expression for the genus two superstring vacuum-to-vacuum amplitude from direct path integral computation. This is a remarkable result. Moreover, they have proved that the amplitude is slice independent, i.e. independent on the parametrization of the even and odd moduli appearing in the path integral. Their solution satisfies the nonrenormalization theorems, as expected in superstring theory. The amplitude is expressed in terms of modular forms of suitable weight defined on a genus two Riemann surface. The measure $d\mu[\Delta]$ appearing in the integral of the amplitude splits in a holomorphic and an antiholomorphic part. This is an essential feature to perform the GSO projection in order to eliminate the tachyon and make the theory stable. All the computations are made explicitly for $g = 2$, but the authors argued that the procedure should work at any genus g .

Following the conjecture (not yet proved) of D'Hoker and Phong one can assume that, as for $g = 2$, the expression of the vacuum-to-vacuum genus g superstring amplitude has the general form:

$$Z^g = \int_{\mathcal{M}_g} (\det \operatorname{Im} \Omega)^{-5} \sum_{\Delta, \Delta'} c_{\Delta \Delta'} d\mu[\Delta](\Omega) \wedge \overline{d\mu[\Delta'](\Omega)} \quad (1.1)$$

with

$$d\mu[\Delta](\Omega) = d\mu_{BOS}(\Omega) \Xi_8(\Delta),$$

$$\Delta = \begin{bmatrix} a \\ b \end{bmatrix}, \quad a, b \in \mathbb{Z}_2^g,$$

$d\mu_{BOS}(\Omega)$ being the well defined bosonic measure, $\Xi_8[\Delta](\Omega)$ are suitable equivariant modular forms, $c_{\Delta \Delta'}$ are phases realizing the opportune GSO projection, Ω is the period matrix of the genus g Riemann surface and M_g is its moduli space. Note that the measure splits in holomorphic and antiholomorphic part $d\mu[\Delta](\Omega)$ and $\overline{d\mu[\Delta'](\Omega)}$ respectively. As discussed by Morozov, there are two different approaches to deal with the superstring measure. The first attitude is to try to prove the general position (1.1) by starting from the Polyakov's measure for NSR string for a fixed characteristic Δ and integrating out the odd moduli. In this way one would directly obtain the explicit expressions for $d\mu[\Delta]$. This requires to realize the holomorphic factorization. In this procedure there are several subtle points, not yet solved, due to the dependence of the result on the choice of the parametrization of even and odd moduli. The second viewpoint is to assume the validity of (1.1), and, from general considerations, to make a reasonable guess for the measure $d\mu[\Delta]$ and use its proprieties to determine it explicitly. In [6, 7] it was adopted this second approach, and, by a slightly modification of the ansätze of D'Hoker and Phong [15, 16] for the properties of $d\mu$, an expression for the superstring chiral measure was found for $g \leq 4$. In the same spirit, in [19] it has been proposed a candidate for the g loop measure, but, due to the presence of square and higher roots, it may be not well defined for $g \geq 5$. Indeed, Salvati Manni in [25] discusses the case $g = 5$ and shows that all the functions appearing in the expression of the (candidate) five loop measure are well defined, at least on the moduli space of curves. In [9] another expression for the five loop superstring measure was proposed using the classical theta constants in which does not appear any root. In all these constructions the forms $\Xi_8[\Delta]$ are built up by making use of the classical theta constants with characteristic. Following this approach one can state a general guess for the supersymmetric invariant measure at any genera. Indeed, we require some constraints for the functions $\Xi_8[\Delta^{(g)}](\Omega^{(g)})$, which consist in three items (see, for details, [5, 6]):

- i. The functions $\Xi_8^{(g)}[\Delta^{(g)}](\Omega^{(g)})$ are holomorphic on the Siegel upper halfplane¹ \mathbb{H}_g (regularity constraint).
- ii. Under the action of the symplectic group $Sp(2g, \mathbb{Z})$ on \mathbb{H}_g , they should transform as follows (transformation constraint):

$$\Xi_8^{(g)}[M \cdot \Delta^{(g)}](M \cdot \Omega) = \det(C\Omega + D)^8 \Xi_8^{(g)}[\Delta^{(g)}](\Omega), \quad (1.2)$$

for all $M \in Sp(2g, \mathbb{Z})$. Here, the affine action of M on the characteristic $\Delta^{(g)}$ is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} := \begin{bmatrix} c \\ d \end{bmatrix}, \quad \begin{pmatrix} t_c \\ t_d \end{pmatrix} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{pmatrix} t_a \\ t_b \end{pmatrix} + \begin{pmatrix} (C^t D)_0 \\ (A^t B)_0 \end{pmatrix} \pmod{2} \quad (1.3)$$

where $N_0 = (N_{11}, \dots, N_{gg})$ is the diagonal of the matrix N .

¹In the following we will not always write the dependence of $\Xi_8^{(g)}$ on $\Omega^{(g)}$.

- iii. The restriction of these functions to “reducible” period matrices is a product of the corresponding functions in lower genus (factorization constraint). More precisely, let

$$D_{k,g-k} := \left\{ \Omega_{k,g-k} := \begin{pmatrix} \Omega_k & 0 \\ 0 & \Omega_{g-k} \end{pmatrix} \in \mathbb{H}_g : \Omega_k \in \mathbb{H}_k, \Omega_{g-k} \in \mathbb{H}_{g-k} \right\} \cong \mathbb{H}_k \times \mathbb{H}_{g-k}.$$

Then we require that for all k , $0 < k < g$,

$$\Xi_8^{(g)} \begin{bmatrix} a_1 \dots a_k & a_{k+1} \dots a_g \\ b_1 \dots b_k & b_{k+1} \dots b_g \end{bmatrix} (\Omega_{k,g-k}) = \Xi_8^{(k)} \begin{bmatrix} a_1 \dots a_k \\ b_1 \dots b_k \end{bmatrix} (\Omega_k) \Xi_8^{(g-k)} \begin{bmatrix} a_{k+1} \dots a_g \\ b_{k+1} \dots b_g \end{bmatrix} (\Omega_{g-k})$$

for all even characteristics $\Delta^{(g)} = \begin{bmatrix} a_1 \dots a_g \\ b_1 \dots b_g \end{bmatrix}$ and all $\Omega_{k,g-k} \in D_{k,g-k}$.

One would hope that these constraints characterize uniquely the measure. Indeed, in [6, 7, 24] the uniqueness of the form $\Xi_8^{(g)}[\Delta^{(g)}]$ has been proved for $g \leq 4$.

In [24] another candidate for the genus five superstring measure has been proposed. The authors have made use of the notion of lattice theta series, see Section 2. The forms $\Xi_8[\Delta]$ defined there to build up the measure, just like the ones obtained using the classical theta constants, satisfies all the constraints. The same formalism has been used to obtain the expressions of the measures for $g \leq 4$. It is not clear if the two constructions are equivalent and lead to the same forms $\Xi_8[\Delta]$, thereby to the same measure $d\mu[\Delta]$. Obviously, this is the case for $g \leq 4$, as a consequence of the uniqueness theorems in lower genus. The goal of the present paper is to show that also in genus five the two constructions are equivalent, and the forms obtained are equal on the whole Siegel half upper plane \mathbb{H}_5 , provided we add to the three constraints the supplementary request of vanishing cosmological constant. Otherwise they could differ for a multiple of the Schottky form $J^{(5)}$ (that vanishes on the locus of trigonal curves, cf. [20]). Actually, adding a scalar multiple of $J^{(5)}$ to a form satisfying the three constraints one obtains a function again satisfying the same constraints: the Schottky $J^{(5)}$ is a modular form of weight eight and the restriction to $\mathbb{H}_1 \times \mathbb{H}_4$ is proportional to $F_{16}^{(1)}$ times $J^{(4)}$ and this product vanishes on the Jacobi locus. This is a remarkable fact because, differently from the genus four case, the zero locus $J^{(5)}$ is not the whole Jacobi locus, but the space of trigonal curves. Thus, the three constraints do not characterize uniquely the superstring measure, see [17, 20, 21]. This freedom can be fixed requiring the vanishing of the cosmological constant. Nevertheless, this should be a prediction of the theory and it should not be imposed by hand. This is a remarkable result both for the viewpoint of physics and of mathematics. Indeed, this shows that there are an infinity of different forms satisfying the three constraints on \mathbb{H}_5 , actually on J_5 . Thus, the constraints, without the additional request on the cosmological constant, do not suffice to characterize the measure uniquely in any genus. Furthermore, a deeper question arises about the conjecture by D'Hoker and Phong on the general expression (1.1) for the superstring chiral measure and about the procedure leading to it. These issues are at the basis of the mathematical correct formulation of the string theory in the perturbative approach. To solve these problems some more insight in the physics leading to the (conjectured) ansatz (1.1) is necessary.

Mathematically, to prove the equivalence of the forms $\Xi_8^{(5)}$, one has to show that the space spanned by the lattice theta series and the one spanned by the eight functions defined in Section 3 (that are a basis for $M_8^\theta(\Gamma_5(2))$, cf. [9]) are the same space of dimension eight. This is the content of the following theorem:

Theorem 1 *The spaces $M_8^\theta(\Gamma_5(2))^{O^+}$ and $M_8^{\Theta_s}(\Gamma_5(2))$ coincide.*

Here $M_8^\theta(\Gamma_5(2))^{O^+}$ is the space of genus five modular forms of weight eight with respect to the group $\Gamma_5(2)$ that are O^+ -invariant polynomials in the classical theta constants, $M_8^{\Theta_s}(\Gamma_5(2))$ is the space of modular forms of weight eight spanned by the lattice theta series ($[\Gamma_5^{\Theta_s}(1, 2), 8]$ in the notation of [24]), and $M_8(\Gamma_5(2))^{O^+}$ is the space of genus five modular forms of weight eight with

respect to the group $\Gamma_5(2)$, which are O^+ -invariant, cf. [5, 9, 10, 21, 24] for details. The theorem follows from a result of Salvati-Manni² [26–28] in which it was proved that the space generated by the lattice theta series contains the subspace generated by classical theta constants that are Γ_g invariant whenever 4 divides the weight (see also [18], theorem VI.1.5). The result applies also for the $\Gamma_g(1, 2)$ case and, as a consequence, one has:

$$M_{4k}^\theta(\Gamma_g(2))^{O^+} \subset M^{\Theta_S}(\Gamma_g(2)), \quad (1.4)$$

for integer k . In genus five the dimensions of both spaces is eight, see [24] for the $M^{\Theta_S}(\Gamma_g(2))$ case, and [9] for the $M_8^\theta(\Gamma_g(2))^{O^+}$ one where also a basis for this space has been constructed. Thus, the theorem follows from the equality of the dimensions of the spaces.

In this paper we exhibit a complete map between the two spaces obtaining all the linear relations between the lattice theta series and the basis functions of the space $M_8^\theta(\Gamma_5(2))^{O^+}$ defined in Section 3 by means of the classical theta constants. To obtain the map we compute certain Fourier coefficients of the functions appearing in the definition of the superstring measure. Since the spaces $M_8^\theta(\Gamma_5(2))^{O^+}$ and $M_8^{\Theta_S}(\Gamma_5(2))$ have dimension eight (see [9, 24]) we need at least eight suitable Fourier coefficients to get linear isomorphisms between these spaces and two copies of \mathbb{C}^8 . In particular, being the two spaces the same, there must be linear relations among the Fourier coefficients of the elements of the two bases, which obviously extend to the complete series. In Section 6.2 we also give an analytic proof of the equivalence between the functions $\Xi_8[\Delta]$ constructed employing the three constraints and the supplementary request of the vanishing of the cosmological constant. In addition, the Fourier coefficients method will permit to obtain, for $g \leq 4$, the complete set of linear relations between the lattice theta series and the basis functions of $M_8^\theta(\Gamma_g(2))^{O^+}$. We will also check the well known linear relations among the lattice theta series themselves [17, 24].

2. Lattice theta series

2.1 Lattices and theta series

In this section we review the notion of lattices, quadratic forms associated with them and lattice theta series, see [1, 8] for details. An n dimensional lattice in \mathbb{R}^n has the form $\Lambda = \{\sum_{i=1}^n a_i v_i \text{ s.t. } a_i \in \mathbb{Z}\}$, where v_i are the elements of a basis of \mathbb{R}^n and are called basis for the lattice. A fundamental region is a building block which when repeated many times fills the whole space with just one lattice point in each copy. Different basis vector could define the same lattice, but the volume of the fundamental region is uniquely determined by Λ . The square of this volume is called the determinant or discriminant of the lattice. The matrix

$$M = \begin{pmatrix} v_{11} & \cdots & v_{1m} \\ \vdots & & \vdots \\ v_{n1} & \cdots & v_{nm} \end{pmatrix}, \quad (2.1)$$

where $v_i = (v_{i1}, \dots, v_{im})$ are the basis vectors is called generator matrix for the lattice. The matrix $A = M^t M$ is called Gram matrix and the entry (i, j) of A is the inner product $v_i \cdot v_j$. The determinant of Λ is the determinant of A . A generic vector $x = (x_1, \dots, x_n)$ of the lattice can be written as $x = \zeta M = \zeta_1 v_1 + \dots + \zeta_n v_n$, where $\zeta = (\zeta_1, \dots, \zeta_n)$ is an arbitrary vector with integer components. Its norm is $N(x) = x \cdot x = \zeta A^t \zeta$. This is a quadratic form associated with the lattice in the integer variables ζ_1, \dots, ζ_n . Any n -dimensional lattice Λ has a dual lattice, Λ^* , given by:

$$\Lambda^* = \{x \in \mathbb{R}^n \text{ s.t. } : x \cdot u \in \mathbb{Z} \text{ for all } u \in \Lambda\}. \quad (2.2)$$

²We are grateful to Riccardo Salvati-Manni who has put his papers to our attention, and explained his main theorem to us.

If a lattice can be obtained from another one by a rotation, reflection and change of scale we say that the two lattices are equivalent (or similar). Two generators matrices define equivalent lattices if and only if they are related by $M' = cUMB$, where c is a non zero constant, U is a matrix with integer entries and determinant ± 1 , and B is a real orthogonal matrix. Then, the corresponding Gram matrices are related by $A' = c^2UA^tU$. If $c = 1$ the two lattices are congruent and if also $\det U = 1$ they are directly congruent. Quadratic forms corresponding to congruent lattices are called integrally equivalent, so there is a one to one correspondence between congruence classes of lattice and integral equivalence classes of quadratic forms. If Λ is a lattice in n -dimensional space that is spanned by n independent vectors (i.e. a full rank lattice), then M has rank n , A is a positive definite matrix, and the associated quadratic form is called a positive definite form. A lattice or a quadratic form is called integral if the inner product of any two lattice vectors is an integer or, equivalently, if the Gram matrix A has integer entries. One can prove that a lattice is integral if and only if $\Lambda \subseteq \Lambda^*$. An integral lattice with $\det \Lambda = 1$, or equivalently with $\Lambda = \Lambda^*$ is called unimodular or self-dual. If Λ is integral then the inner product $x \cdot x$ is necessarily an integer for all points x of the lattice. If $x \cdot x$ is an even integer for all $x \in \Lambda$ then the lattice is called even, otherwise odd. Even unimodular lattices exist if and only if the dimension is a multiple of 8, while odd unimodular lattices exist in all dimensions.

For a lattice Λ let N_m be the number of vectors $x \in \Lambda$ of norm $m = x \cdot x$. Thus, N_m is also the number of integral vectors ζ that are solutions of the Diophantine equation

$$\zeta A^t \zeta = m \quad (2.3)$$

or, in other words, the number of times that the quadratic form associated with Λ represents the number m . The (genus one) theta series of a lattice Λ is a holomorphic function on the Siegel upper half space \mathbb{H}_1 , defined by

$$\Theta_\Lambda(\tau) = \sum_{x \in \Lambda} q^{x \cdot x} = \sum_{m=0}^{\infty} N_m q^m, \quad (2.4)$$

where $q = e^{\pi i \tau}$ and $\tau \in \mathbb{H}_1$. For example, the theta series associated to the lattice \mathbb{Z} is the classical Jacobi theta constant $\Theta_{\mathbb{Z}}(\tau) = \sum_{m=-\infty}^{\infty} q^{m^2} = 1 + 2q + 2q^4 + 2q^9 + \dots \equiv \theta_{[0]}^{[0]}(\tau)$, see Section 3. This definition generalizes to theta series of arbitrary genus g . In this case the vector ζ becomes a $g \times n$ matrix $\underline{\zeta}$ with integer entries. In addition, one also introduces a $g \times n$ array \underline{x} whose rows are the vectors of the lattice Λ . It can be written as $\underline{x} = \underline{\zeta} M$. Let $N_{\underline{m}} \in \mathbb{Z}$ be the number of integral matrix solutions of the Diophantine system

$$\underline{\zeta} A^t \underline{\zeta} = \underline{m}, \quad (2.5)$$

where \underline{m} is a $g \times g$ symmetric matrix with integer entries. The component (i, j) of \underline{m} represents the scalar product between the vectors $x_i \in \Lambda$ and $x_j \in \Lambda$ of \underline{x} . Thus, $N_{\underline{m}}$ is also the number of the sets \underline{x} of g -vectors such that $x_i \cdot x_j = m_{ij}$. In the same spirit of the genus one case, the genus g theta series associated to a lattice Λ is a holomorphic function on the Siegel upper half space \mathbb{H}_g , defined by

$$\Theta_\Lambda^{(g)}(\tau) = \sum_{x \in \Lambda^{(g)}} e^{\pi i \text{Tr}(x \cdot x)} = \sum_{\underline{\zeta} \in \mathbb{Z}^{g,n}} e^{\pi i \text{Tr}(\underline{\zeta} A^t \underline{\zeta} \tau)} = \sum_{\underline{m}} N_{\underline{m}} \prod_{i \leq j} e^{\pi i m_{ij} \tau_{ij}}, \quad (2.6)$$

and $\tau \in \mathbb{H}_g$. Lattice theta series corresponding to a self-dual n -dimensional lattice, with n divisible by 8, is a modular form of weight $\frac{n}{2}$ with respect to the group $\Gamma_g(1, 2)$ if the lattice is odd and with respect to Γ_g if the lattice is even³. Thus, lattice theta series associated to 16-dimensional self-dual

³We recall the definitions:

$$\begin{aligned} \Gamma_g &:= \text{Sp}(2g, \mathbb{Z}), & \Gamma_g(2) &:= \{M \in \Gamma_g \mid M \equiv \mathbb{I} \pmod{2}\} \\ \Gamma_g(1, 2) &:= \{M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g(2) \mid A^t B = \text{diag} C^t D = 0 \pmod{2}\}. \end{aligned}$$

lattices are modular forms of weight 8. There are eight 16-dimensional self-dual lattice [8], two even and six odd, and they can be obtained from the root lattice of some Lie algebra. See also [17,21,24]. In what follows we will use a nice property of lattice theta series when restricted to block diagonal period matrices: indeed, they factorize in a very simple way when $\tau \in \mathbb{H}_k \times \mathbb{H}_{g-k}$:

$$\Theta_{\Lambda}^{(g)} \begin{pmatrix} \tau_k & 0 \\ 0 & \tau_{g-k} \end{pmatrix} = \Theta_{\Lambda}^{(k)}(\tau_k) \Theta_{\Lambda}^{(g-k)}(\tau_{g-k}). \quad (2.7)$$

2.2 Fourier coefficients of lattice theta series

In order to express the relations between lattice theta series and the classical theta constants, we first expand in Fourier series the lattice theta constants. We just need the coefficient $N_{\underline{m}}$ of the series (2.6) for some integer matrix \underline{m} . It is known (cf. [24]) that in genus five the eight theta series are all independent, whereas for lower genus there are linear relations among them. Thus, we have to choose at least eight \underline{m} in such a way that the matrix of the Fourier coefficients $N_{\underline{m}}$ of the eight theta series has rank 8. In Table 1 are shown the Fourier coefficients for the eight theta series up to $g = 5$. We computed the coefficients for the matrices:

$$\begin{aligned} m_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & m_2 &= \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & m_3 &= \begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & m_4 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & m_5 &= \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ m_6 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & m_7 &= \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & m_8 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & m_9 &= \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} & m_{10} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Appealing to the geometric interpretation for the matrices \underline{m}_k , with $k = 1, \dots, 10$, for each of the eight 16-dimensional self-dual lattices, we are looking for the number of integer solutions of the Diophantine equation $\zeta A_{\Lambda} \zeta^t = \underline{m}_k$. In other terms, we are counting the number of sets \underline{x} of five vectors in the lattice Λ such that the vector x_i has norm $(m_k)_{ii}$ and the inner product with the vector x_j is $x_i \cdot x_j = (m_k)_{ij}$. It is clear that the Fourier coefficients corresponding, for example, to the matrix m_4 can be interpreted as the Fourier coefficients for the genus two theta series in which the two orthogonal vectors x_1 and x_2 have both norm 1, but also as the coefficients of the theta series of genus $g > 2$ in which the vectors x_i with $i > 2$ have null norm. It is not hard to perform this computation using a software like Magma, although the computation of the coefficients corresponding to the matrix $\text{diag}(2, 2, 2, 2, 0)$ may take some hours.

The notation of the Table 1 is the same as in [24]. The rows contain the Fourier coefficients of the theta series corresponding to the eight lattices $(D_8 \oplus D_8)^+$, $\mathbb{Z} \oplus A_{15}^+$, $\mathbb{Z}^2 \oplus (E_7 \oplus E_7)^+$, $\mathbb{Z}^4 \oplus D_{12}^+$, $\mathbb{Z}^8 \oplus E_8$, \mathbb{Z}^{16} , $E_8 \oplus E_8$, and D_{16}^+ , where the last two lattices are the even ones. At the top of the columns we just indicated the diagonal elements of the matrices \underline{m}_k , the other elements being zero. As anticipated, the rank of the full matrix of the coefficient is eight, thus no linear relations between genus five theta series exist. However, considering the same matrix for genus less than five one can obtain the relations between theta series, as we will show in the following, for every $g \leq 4$. In the top of the Table we write in bold the matrices strictly necessary for the computation, whereas some other columns are added as a check. The same convention will be used throughout in the paper.

3. Riemann theta constants and the forms $\Xi_8^{(g)}$

The form $\Xi_8^{(g)}[0^{(g)}]$, appearing in the expression for the superstring chiral measure, belongs to $M_8(\Gamma_g(2))^{O^+}$, the space of modular forms of weight eight with respect to the group $\Gamma_g(2)$, and invariant under the action of $O^+ := \Gamma_g(1, 2)/\Gamma_g(2)$, see [9, 10] for details. In [9] a basis for these

	(1, 0, 0, 0, 0)	(2, 0, 0, 0, 0)	(3, 0, 0, 0, 0)	(1, 1, 0, 0, 0)	(2, 2, 0, 0, 0)	(1, 1, 1, 0, 0)	(2, 2, 2, 0, 0)	(1, 1, 1, 1, 0)	(2, 2, 2, 2, 0)	(1, 1, 1, 1, 1)
$\Theta_{(D_8 \oplus D_8)^+}$	0	224	4096	0	38976	0	5069568	0	475270656	0
$\Theta_{\mathbb{Z} \oplus A_{15}^+}$	2	240	4120	0	43680	0	5765760	0	518918400	0
$\Theta_{\mathbb{Z}^2 \oplus (E_7 \oplus E_7)^+}$	4	256	4144	8	48896	0	6676992	0	644668416	0
$\Theta_{\mathbb{Z}^4 \oplus D_{12}^+}$	8	288	4192	48	60864	192	9181440	384	964200960	0
$\Theta_{\mathbb{Z}^8 \oplus E_8}$	16	352	4288	224	90944	2688	17176320	26880	2316142080	215040
$\Theta_{\mathbb{Z}^{16}}$	32	480	4480	960	175680	26880	47174400	698880	8858304000	16773120
$\Theta_{E_8 \oplus E_8}$	0	480	0	0	175680	0	47174400	0	9064742400	0
$\Theta_{D_{16}^+}$	0	480	0	0	175680	0	47174400	0	8858304000	0

Table 1: Fourier coefficients for the lattice theta series.

spaces has been found for $g \leq 5$ and a suitable linear combination among these basis vectors has been obtained by imposing the constraints of Section 1.

Let us now discuss the theta constants with characteristics, which are a powerful tool for constructing modular forms on $\Gamma_g(2)$. An even characteristic is a $2 \times g$ matrix $\Delta = \begin{bmatrix} a \\ b \end{bmatrix}$, with $a, b \in \{0, 1\}$ and $\sum a_i b_i \equiv 0 \pmod{2}$. Let $\tau \in \mathbb{H}_g$, the Siegel upper half space, then we define the theta constants with characteristic:

$$\theta_{[b]}^a(\tau) := \sum_{t \in \mathbb{Z}^g} e^{\pi i((m+a/2)\tau^t(m+a/2)+(m+a/2)^t b)}, \quad (3.1)$$

where m is a row vector. Thus, theta constants are holomorphic functions on \mathbb{H}_g . One can build modular forms of weight eight as suitable polynomials of degree sixteen in the theta constants, see [5, 9, 10] for details. Defining the $g \times g$ symmetric matrix M with entries $M_{ii} = m_i^2 + a_i m_i + \frac{a_i^2}{4}$, $i = 1, \dots, g$ and $M_{ij} = m_i m_j + \frac{a_j}{2} m_i + \frac{a_i}{2} m_j + \frac{a_i a_j}{4}$, $1 \leq i < j \leq g$, the definition of theta constant can be rewritten as

$$\begin{aligned} \theta_{[b]}^a(\tau) &:= \sum_{m \in \mathbb{Z}^g} (-)^{\frac{a_1 b_1}{2} + \dots + \frac{a_g b_g}{2}} (-)^{b_1 m_1 + \dots + b_g m_g} e^{\pi i \text{Tr}(M\tau)} \\ &= (-)^{\frac{a_1 b_1}{2} + \dots + \frac{a_g b_g}{2}} \sum_{m \in \mathbb{Z}^g} (-)^{b_1 m_1 + \dots + b_g m_g} \prod_{i \leq j} e^{\pi i (2 - \delta_{ij}) M_{ij} \tau_{ij}} \\ &= \sum_{A \in \frac{\mathbb{Z}^{g,g}}{4}, tA=A} N_A \prod_{i \leq j} e^{\pi i A_{ij} \tau_{ij}}, \end{aligned} \quad (3.2)$$

where A is a symmetric $g \times g$ matrix with entries in $\frac{1}{4}\mathbb{Z}$ and N_A is an integer coefficient. In particular N_A is the number of times⁴ that the particular matrix A appears in the sum (3.2). Note that the factor $(-)^{\frac{a_1 b_1}{2} + \dots + \frac{a_g b_g}{2}}$ is a global sign depending only on the characteristic Δ and the coefficient $(-)^{b_1 m_1 + \dots + b_g m_g}$ is a sign depending on the second row of the theta characteristic and on the matrix M .

In [9] there have been computed the dimensions of the spaces of O^+ -invariants for $g \leq 5$. It turned out that these dimensions are 3, 4, 5, 7 and 8 for $g = 1, 2, 3, 4$ and 5 respectively. Moreover, a basis has been provided for each of these genera, by means of the classical Riemann theta constants. For each genus one defines:

$$\begin{aligned} F_1^{(g)} &:= \theta[0^{(g)}]^{16}, & F_8^{(g)} &:= \left(\sum_{\Delta^{(g)}} \theta[\Delta^{(g)}] \right)^8, \\ F_2^{(g)} &:= \theta[0^{(g)}]^4 \sum_{\Delta^{(g)}} \theta[\Delta^{(g)}]^{12}, & F_{88}^{(g)} &:= \sum_{(\Delta_i^{(g)}, \Delta_j^{(g)})_o} \theta[\Delta_i^{(g)}]^8 \theta[\Delta_j^{(g)}]^8, \\ F_3^{(g)} &:= \theta[0^{(g)}]^8 \sum_{\Delta^{(g)}} \theta[\Delta^{(g)}]^8, & F_{16}^{(g)} &:= \sum_{\Delta^{(g)}} \theta[\Delta^{(g)}]^{16}, \end{aligned}$$

⁴Counted with signs given by the factor multiplying the product of exponentials.

where $(\Delta_i^{(g)}, \Delta_j^{(g)})_o$ stands for the set of all pairs of distinct even characteristics whose sum is odd. Behind these, in [9] are defined the forms $G_3^{(g)}[0^{(g)}]$ for $g = 4, 5$ and $G_4^{(g)}[0^{(g)}]$ for $g = 5$. However, we note that $G_3^{(g)}[0^{(g)}]$ ($G_4^{(g)}[0^{(g)}]$) could be defined for every genus $g \geq 3$ ($g \geq 4$) considering three (four) dimensional isotropic subspace of \mathbf{F}_2^{2g} , where \mathbf{F}_2 is the field of two elements. See [6, 7, 9, 19] for more definitions and details. Consider the form $J^{(g)} := 2^g \sum_{\Delta^{(g)}} \theta[\Delta^{(g)}]^{16} - (\sum_{\Delta^{(g)}} \theta[\Delta^{(g)}]^8)^2 = 2^g F_{16}^{(g)} - F_8^{(g)}$. It vanishes identically in genus $g \leq 2$, for $g = 3$ vanishes on the whole \mathbb{H}_3 , for $g = 4$ on the Jacobi locus and for $g = 5$ on the locus of trigonal curves. Clearly all these functions are not linear independent for $g < 5$, thus for each genus we extract a basis as reported in Table 2, where the symbol \checkmark means that the same function as in lower genus has been taken as element of the basis (with obvious modifications). We will indicate generically with $e_i^{(g)}$ the elements of

Basis/ g	1	2	3	4	5
F_1	$\theta[0]^{16}$	\checkmark	\checkmark	\checkmark	\checkmark
F_2	$\theta[0]^4 \sum_{\Delta} \theta[\Delta]^{12}$	\checkmark	\checkmark	\checkmark	\checkmark
F_{16}	$\sum_{\Delta} \theta[\Delta]^{16}$	\checkmark	\checkmark	\checkmark	\checkmark
F_3		$\theta[0]^8 \sum_{\Delta} \theta[\Delta]^8$	\checkmark	\checkmark	\checkmark
F_{88}			$\sum_{(\Delta_i, \Delta_j)_o} \theta[\Delta_i]^8 \theta[\Delta_j]^8$	\checkmark	\checkmark
F_8				$(\sum_{\Delta} \theta[\Delta]^8)^2$	\checkmark
$G_3[0]$				$G_3[0]$	\checkmark
$G_4[0]$					$G_4[0]$

Table 2: Basis for the O^+ -invariants

the genus g basis. Each function in Table 2 is a suitable polynomial of degree sixteen in the theta constants and the forms $\Xi_8^{(g)}[0^{(g)}]$ are suitable linear combinations of them. In order to compare the two expressions of the proposed superstring chiral measure for $g \leq 5$ we also need the Fourier coefficients of the basis of the O^+ -invariants. In general, given two series $\sum_n a_n q^n$ and $\sum_m b_m q^m$, their product is $\sum_n a_n q^n \sum_m b_m q^m = \sum_k c_k q^k$, with $c_k = \sum_{m+n=k} a_n b_m$. In this way one computes the Fourier coefficients of the eight functions starting from the ones of the theta constants. However, for increasing g the computation becomes extremely lengthy, due to the huge number of monomials appearing in the definition of the $e_i^{(g)}$. Thus, although in principle possible by hand, we perform the computation using the C++ programming language, see appendix A.

4. CDG ansätze and OPSY ansätze for $\Xi_8^{(g)}$

Before starting the computation of the Fourier coefficients we review the expressions of the forms $\Xi_8^{(g)}[0^{(g)}]$ for $g \leq 5$ in both formalisms. In what follows we will call $\Xi_8^{(g)}[0]_{CDG}$ the forms defined in [6, 7, 9] (even though we will use the particular basis of [9]) and $\Xi_8^{(g)}[0]_{OPSY}$ the forms of [24]. The expressions of the forms $\Xi_8^{(g)}[0^{(g)}]$ found in [9] using the classical theta constants, see also [6, 7]

for the case $g \leq 4$, are:

$$\begin{aligned}
\Xi_8^{(1)}[0]_{CDG} &= \frac{2}{3}F_1^{(1)} - \frac{1}{3}F_2^{(1)}, \\
\Xi_8^{(2)}[0]_{CDG} &= \frac{2}{3}F_1^{(2)} + \frac{1}{3}F_2^{(2)} - \frac{1}{2}F_3^{(2)}, \\
\Xi_8^{(3)}[0]_{CDG} &= \frac{1}{3}F_1^{(3)} + \frac{1}{3}F_2^{(3)} - \frac{1}{4}F_3^{(3)} - \frac{1}{64}F_8^{(3)} + \frac{1}{16}F_{88}^{(3)}, \\
\Xi_8^{(4)}[0]_{CDG} &= \frac{1}{6}F_1^{(4)} + \frac{1}{3}F_2^{(4)} - \frac{1}{8}F_3^{(4)} + \frac{1}{64}F_8^{(4)} - \frac{1}{16}F_{88}^{(4)} - \frac{1}{2}G_3^{(4)}[0^{(4)}] - c_4 J^{(4)}, \\
\Xi_8^{(5)}[0]_{CDG} &= \frac{1}{12}F_1^{(5)} + \frac{1}{3}F_2^{(5)} - \frac{1}{16}F_3^{(5)} - \frac{1}{32}F_8^{(5)} + \frac{1}{8}F_{88}^{(5)} - \frac{1}{4}G_3^{(5)}[0^{(5)}] + 2G_4^{(5)}[0^{(5)}] - c_5 J^{(5)}.
\end{aligned}$$

Here we have included the terms $-c_4 J^{(4)}$ and $-c_5 J^{(5)}$ to have vanishing cosmological constant on the whole \mathbb{H}_4 and \mathbb{H}_5 and to compare these functions to the ones of [24]. In particular, $c_4 = \frac{3^2 \cdot 5}{2^6 \cdot 7 \cdot 17}$ and $c_5 = \frac{17}{2^5 \cdot 7 \cdot 11}$ (see [9]). The forms $\Xi_8^{(g)}[0^{(g)}]$ defined in [24] by means of the lattice theta series are:

$$\begin{aligned}
\Xi_8^{(1)}[0]_{OPSY} &= -\frac{31}{32}\Theta_{(D_8 \oplus D_8)^+} + \frac{512}{315}\Theta_{\mathbb{Z} \oplus A_{15}^+} - \frac{16}{21}\Theta_{\mathbb{Z}^2 \oplus (E_7 \oplus E_7)^+} + \frac{1}{9}\Theta_{\mathbb{Z}^4 \oplus D_{12}^+} - \frac{1}{168}\Theta_{\mathbb{Z}^8 \oplus E_8} \\
&\quad + \frac{1}{10080}\Theta_{\mathbb{Z}^{16}}, \\
\Xi_8^{(2)}[0]_{OPSY} &= \frac{155}{512}\Theta_{(D_8 \oplus D_8)^+} - \frac{16}{21}\Theta_{\mathbb{Z} \oplus A_{15}^+} + \frac{23}{42}\Theta_{\mathbb{Z}^2 \oplus (E_7 \oplus E_7)^+} - \frac{3}{32}\Theta_{\mathbb{Z}^4 \oplus D_{12}^+} + \frac{29}{5376}\Theta_{\mathbb{Z}^8 \oplus E_8} \\
&\quad - \frac{1}{10752}\Theta_{\mathbb{Z}^{16}}, \\
\Xi_8^{(3)}[0]_{OPSY} &= -\frac{155}{4096}\Theta_{(D_8 \oplus D_8)^+} + \frac{1}{9}\Theta_{\mathbb{Z} \oplus A_{15}^+} - \frac{3}{32}\Theta_{\mathbb{Z}^2 \oplus (E_7 \oplus E_7)^+} + \frac{101}{4608}\Theta_{\mathbb{Z}^4 \oplus D_{12}^+} - \frac{3}{2048}\Theta_{\mathbb{Z}^8 \oplus E_8} \\
&\quad + \frac{1}{36864}\Theta_{\mathbb{Z}^{16}}, \\
\Xi_8^{(4)}[0]_{OPSY} &= \frac{31}{16384}\Theta_{(D_8 \oplus D_8)^+} - \frac{1}{168}\Theta_{\mathbb{Z} \oplus A_{15}^+} + \frac{29}{5376}\Theta_{\mathbb{Z}^2 \oplus (E_7 \oplus E_7)^+} - \frac{3}{2048}\Theta_{\mathbb{Z}^4 \oplus D_{12}^+} \\
&\quad + \frac{23}{172032}\Theta_{\mathbb{Z}^8 \oplus E_8} - \frac{1}{344064}\Theta_{\mathbb{Z}^{16}} - b_4 \left(\Theta_{E_8 \oplus E_8} - \Theta_{D_{16}^+} \right), \\
\Xi_8^{(5)}[0]_{OPSY} &= -\frac{1}{32768}\Theta_{(D_8 \oplus D_8)^+} + \frac{1}{10080}\Theta_{\mathbb{Z} \oplus A_{15}^+} - \frac{1}{10752}\Theta_{\mathbb{Z}^2 \oplus (E_7 \oplus E_7)^+} + \frac{1}{36864}\Theta_{\mathbb{Z}^4 \oplus D_{12}^+} \\
&\quad - \frac{1}{344064}\Theta_{\mathbb{Z}^8 \oplus E_8} + \frac{1}{10321920}\Theta_{\mathbb{Z}^{16}} - b_5 \left(\Theta_{E_8 \oplus E_8} - \Theta_{D_{16}^+} \right).
\end{aligned}$$

Here $b_4 = \frac{2^2 \cdot 3^3 \cdot 5 \cdot 11}{7 \cdot 17}$ and $b_5 = -\frac{2^5 \cdot 17}{7 \cdot 11}$ (see [21]) make the cosmological constant vanishing on the whole \mathbb{H}_4 and \mathbb{H}_5 respectively. One of the goals of this paper is to show that up to genus five the two expressions for the superstring chiral measure coincide. For $g \leq 4$ this was expected from the uniqueness theorems proved in [6, 10, 24]. Instead, for $g = 5$ the formalism of the classical theta constants and the one of the lattice theta series lead to distinct functions both satisfying the three constraints of Section 1. Actually, this indetermination could appear for each choice for the basis of the spaces $M_8^g(\Gamma_5(2))^{O^+}$ or $M_8^{\Theta_S}(\Gamma_5(2))$. Moreover, their difference is proportional to the Schottky form $J^{(5)}$ and the two forms become equivalent if one requires also the vanishing of the cosmological constants, i.e. the vanishing of their sum over all the even characteristics, $\sum_{\Delta} \Xi_8^{(g)}[\Delta^{(g)}] = 0$. The forms $\Xi_8^{(g)}[\Delta^{(g)}]$ are obtained from the $\Xi_8^{(g)}[0^{(g)}]$ by the action of the symplectic group, see [6].

5. Change of basis

In this section we search the relations between the functions defined in Section 3 and the lattice

theta series. For $g \leq 3$ one can proceed in several way, but for $g \geq 4$ the knowledge of the Fourier coefficients becomes necessary.

5.1 The case $g=1$

In genus one we can expand the eight lattice theta series on the basis of O^+ -invariants $F_1^{(1)}$, $F_2^{(1)}$, $F_{16}^{(1)}$ using the Table 2 in [24], page 491, that we reproduce in Table 3. There, Λ_i , $i = 0, \dots, 7$ label

i	Λ_i	τ_i	b_i	c_i
0	$(D_8 \oplus D_8)^+$	0	1	0
1	$\mathbb{Z} \oplus A_{15}^+$	2	1	0
2	$\mathbb{Z}^2 \oplus (E_7 \oplus E_7)^+$	4	1	0
3	$\mathbb{Z}^4 \oplus D_{12}^+$	8	1	0
4	$\mathbb{Z}^8 \oplus E_8$	16	1	0
5	\mathbb{Z}^{16}	32	1	0
6	$E_8 \oplus E_8$	0	0	1
7	D_{16}^+	0	0	1

Table 3: Linear relation between lattice theta series.

the eight lattices and τ_i , b_i and c_i are the coefficients of the linear expansions of the series Θ_{Λ_i} on the basis $\Xi_8^{(1)}[0^{(1)}]_{OPSY}$, $\Theta_{\Lambda_0}^{(1)}$, $\Theta_{\Lambda_6}^{(1)}$ for the space $[\Gamma_1(1, 2), 8]$. Thus, $\Theta_{\Lambda_i}^{(1)} = \tau_i \Xi_8^{(1)}[0^{(1)}] + b_i \Theta_{\Lambda_0}^{(1)} + c_i \Theta_{\Lambda_6}^{(1)}$. It is easy to show that the relations $\Xi_8^{(1)}[0^{(1)}]_{OPSY} = \frac{1}{16} \theta_{[0]}^{[0]^4} \eta^{12} = \frac{1}{24} F_1^{(1)} - \frac{1}{48} F_2^{(1)}$ (cf. [9], section 4.1), $\Theta_{\mathbb{Z}^{16}} = \theta_{[0]}^{[0]^{16}} \equiv F_1^{(1)}$ (cf. [8], first formula, page 46), $\Theta_{(D_8 \oplus D_8)^+} = -\frac{1}{3} F_1^{(1)} + \frac{2}{3} F_2^{(1)}$ (by the fifth line of Table 3) and $\Theta_{E_8 \oplus E_8} = \frac{1}{2} F_{16}^{(1)}$ (cf. [8], last formula, page 47) hold. Thus, the linear relations of Table 4 follow immediately.

Theta series/Basis	F_1	F_2	F_{16}
$\Theta_{(D_8 \oplus D_8)^+}$	-1/3	2/3	0
$\Theta_{\mathbb{Z} \oplus A_{15}^+}$	-1/4	15/24	0
$\Theta_{\mathbb{Z}^2 \oplus (E_7 \oplus E_7)^+}$	-1/6	7/12	0
$\Theta_{\mathbb{Z}^4 \oplus D_{12}^+}$	0	1/2	0
$\Theta_{\mathbb{Z}^8 \oplus E_8}$	1/3	1/3	0
$\Theta_{\mathbb{Z}^{16}}$	1	0	0
$\Theta_{E_8 \oplus E_8}$	0	0	1/2
$\Theta_{D_{16}^+}$	0	0	1/2

Table 4: Theta series on the basis F_1 , F_2 and F_{16} .

Moreover, the lattice theta series in genus one are not all linear independent, but they generate a three dimensional vector space. Therefore, they must satisfy some linear relations, which can be obtained studying the five dimensional kernel of the first three bold columns of Table 1 computed

with Magma. This give the following relations among theta series:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 15 & -16 & 0 & 0 & 0 & 1 & 0 & 0 \\ 7 & -8 & 0 & 0 & 1 & 0 & 0 & 0 \\ 3 & -4 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \Theta_{(D_8 \oplus D_8)^+} \\ \Theta_{\mathbb{Z} \oplus A_{15}^+} \\ \Theta_{\mathbb{Z}^2 \oplus (E_7 \oplus E_7)^+} \\ \Theta_{\mathbb{Z}^4 \oplus D_{12}^+} \\ \Theta_{\mathbb{Z}^8 \oplus E_8} \\ \Theta_{\mathbb{Z}^{16}} \\ \Theta_{E_8 \oplus E_8} \\ \Theta_{D_{16}^+} \end{pmatrix} = \underline{0}.$$

As a check, one can show that these relations are in complete agreement with those that can be computed using Table 4. For example, from the second line one reads $15\Theta_{(D_8 \oplus D_8)^+} - 16\Theta_{\mathbb{Z} \oplus A_{15}^+} + \Theta_{\mathbb{Z}^8 \oplus E_8} = 0$. From the Fourier coefficients of the eight theta series and from their expansion on the basis of the O^+ -invariants we can also find the Fourier coefficients for the three functions $F_1^{(1)}$, $F_2^{(1)}$ and $F_{16}^{(1)}$ expressed as polynomials of degree sixteen in the classical theta constants as showed in Table 5. As this space is three dimensional, we just need three coefficients and we choose the ones corresponding to the matrices (that in $g = 1$ are just numbers) 1, 2 and 3. Using the C++ program (cf. Appendix A) we also checked the correctness of the coefficients and further we computed the coefficient corresponding to the matrix 0. Actually for lower genus this computation can be performed easily by hand.

Functions/ m	0	1	2	3
F_1	1	32	480	4480
F_2	2	16	576	8384
F_{16}	2	0	960	0

Table 5: Fourier coefficients for the F_1 , F_2 and F_{16} in genus one.

5.2 The case $g=2$

Using the factorization properties of the classical theta constants one obtains the factorization of the basis of the space of $O^{(+)}$ invariants (cf. [9], section 4.2), whereas for the theta series one can apply property (2.7). Thus, we can find the expansions of the $g = 2$ theta series on the basis of the four O^+ -invariants as follows (sometimes for brevity we will indicate this space as O_g). In general we have

$$\Theta_{\Lambda_i}^{(g)}(\tau) = \sum_{j=1}^{\dim O_g} k_i^{(g)j} e_j^{(g)}, \quad (5.1)$$

where $e_j^{(g)}$ are the basis for the genus g O^+ -invariants, written as polynomials in the classical theta constants, and $k_i^{(g)j}$ are the constants we want to determine. The restriction on $\mathbb{H}_1 \times \mathbb{H}_{g-1}$ of the theta series is

$$\begin{aligned} \Theta_{\Lambda_i}^{(g)}(\tau_{1,g-1}) &= \Theta_{\Lambda_i}^{(g)}(\tau_1) \Theta_{\Lambda_i}^{(g-1)}(\tau_{g-1}) = \sum_{j=1}^{\dim O_1} k_i^{(1)j} e_j^{(1)} \sum_{m=1}^{\dim O_{g-1}} k_i^{(g-1)m} e_m^{(g-1)} \\ &= \sum_{j=1}^{\dim O_1} \sum_{m=1}^{\dim O_{g-1}} k_i^{(1)j} k_i^{(g-1)m} e_j^{(1)} e_m^{(g-1)}, \end{aligned} \quad (5.2)$$

but also

$$\begin{aligned}
\Theta_{\Lambda_i}^{(g)}(\tau_{1,g-1}) &= \sum_{j=1}^{\dim O_g} k_i^{(g)j} e_j^{(g)}(\tau_{1,g-1}) = \sum_{j=1}^{\dim O_g} k_i^{(g)j} \left(\sum_{l=1}^{\dim O_1} a_j^{(1)l} e_l^{(1)} \right) \left(\sum_{m=1}^{\dim O_{g-1}} a_j^{(g-1)m} e_m^{(g-1)} \right) \\
&= \sum_{j=1}^{\dim O_g} \sum_{l=1}^{\dim O_1} \sum_{m=1}^{\dim O_{g-1}} k_i^{(g)j} a_j^{(1)l} a_j^{(g-1)m} e_l^{(1)} e_m^{(g-1)}. \tag{5.3}
\end{aligned}$$

The expressions (5.2) and (5.3) must be equal. Thus, for every fixed choice of l and m we obtain a linear equation in $k_i^{(g)j}$. The solution of this linear system gives the coefficients in the change of basis. We give the result for the case $g = 2$ in Table 6.

Theta series/Basis	F_1	F_2	F_3	F_{16}
$\Theta_{(D_8 \oplus D_8)^+}$	1/3	2/3	-1/2	0
$\Theta_{\mathbb{Z} \oplus A_{15}^+}$	7/32	35/64	-45/128	0
$\Theta_{\mathbb{Z}^2 \oplus (E_7 \oplus E_7)^+}$	1/8	7/16	-7/32	0
$\Theta_{\mathbb{Z}^4 \oplus D_{12}^+}$	0	1/4	0	0
$\Theta_{\mathbb{Z}^8 \oplus E_8}$	0	0	1/4	0
$\Theta_{\mathbb{Z}^{16}}$	1	0	0	0
$\Theta_{E_8 \oplus E_8}$	0	0	0	1/4
$\Theta_{D_{16}^+}$	0	0	0	1/4

Table 6: Theta series on the basis F_1, F_2, F_3 and F_{16} .

As expected (cf. [9, 10, 24]), the matrix of the coefficients has rank four, which is then also the dimension of the kernel and we can determine the linear relations among the theta series

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ -105 & 224 & -120 & 0 & 0 & 1 & 0 & 0 \\ -21 & 48 & -28 & 0 & 1 & 0 & 0 & 0 \\ -3 & 8 & -6 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \Theta_{(D_8 \oplus D_8)^+} \\ \Theta_{\mathbb{Z} \oplus A_{15}^+} \\ \Theta_{\mathbb{Z}^2 \oplus (E_7 \oplus E_7)^+} \\ \Theta_{\mathbb{Z}^4 \oplus D_{12}^+} \\ \Theta_{\mathbb{Z}^8 \oplus E_8} \\ \Theta_{\mathbb{Z}^{16}} \\ \Theta_{E_8 \oplus E_8} \\ \Theta_{D_{16}^+} \end{pmatrix} = \underline{0}.$$

For example, from the third line, we have $-21 \Theta_{(D_8 \oplus D_8)^+} + 48 \Theta_{\mathbb{Z} \oplus A_{15}^+} - 28 \Theta_{\mathbb{Z}^2 \oplus (E_7 \oplus E_7)^+} + \Theta_{\mathbb{Z}^8 \oplus E_8} = 0$. One can verify that the same relations result by the study of the kernel of the first four bold columns of the Table 1 of the Fourier coefficients for the lattice theta series.

As for the genus one case, we compute the Fourier coefficients for the four functions $F_1^{(2)}, F_2^{(2)}, F_3^{(2)}$ and $F_{16}^{(2)}$ both using the previous results and the **C++** program. The Table 7 shows the result.

5.3 The case $g = 3$

In genus three we can obtain the expansion of the theta series on the basis $e_i^{(3)}$ with the method of factorization explained in the previous section. We report the result in Table 8. As expected, the matrix of the coefficients has rank five, thus its kernel has dimension three. Again we find the

Functions/m	(0, 0)	(1, 0)	(2, 0)	(3, 0)	(1, 1)	(2, 2)
F₁	1	32	480	4480	960	175680
F₂	4	32	1152	16768	192	243456
F₃	4	64	1408	17152	896	363776
F₁₆	4	0	1920	0	0	702720
<i>F₈</i>	16	0	7680	0	0	2810880
<i>F₈₈</i>	0	0	1024	-16384	0	546816

Table 7: Fourier coefficients for the F_1 , F_2 , F_3 and F_{16} in genus two.

Theta series/Basis	F_1	F_2	F_3	F_{16}	F_{88}
$\Theta_{(D_8 \oplus D_8)^+}$	0	0	0	1/8	-1/16
$\Theta_{\mathbb{Z} \oplus A_{15}^+}$	7/512	35/512	-45/2048	315/4096	-315/8192
$\Theta_{\mathbb{Z}^2 \oplus (E_7 \oplus E_7)^+}$	1/64	7/64	-7/256	21/512	-21/1024
$\Theta_{\mathbb{Z}^4 \oplus D_{12}^+}$	0	1/8	0	0	0
$\Theta_{\mathbb{Z}^8 \oplus E_8}$	0	0	1/8	0	0
$\Theta_{\mathbb{Z}^{16}}$	1	0	0	0	0
$\Theta_{E_8 \oplus E_8}$	0	0	0	1/8	0
$\Theta_{D_{16}^+}$	0	0	0	1/8	0

Table 8: Theta series on the basis F_1 , F_2 , F_3 , F_{16} and F_{88} in genus three.

linear relations studying the kernel of the matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 315 & -896 & 720 & -140 & 0 & 1 & 0 & 0 \\ 21 & -64 & 56 & -14 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \Theta_{(D_8 \oplus D_8)^+} \\ \Theta_{\mathbb{Z} \oplus A_{15}^+} \\ \Theta_{\mathbb{Z}^2 \oplus (E_7 \oplus E_7)^+} \\ \Theta_{\mathbb{Z}^4 \oplus D_{12}^+} \\ \Theta_{\mathbb{Z}^8 \oplus E_8} \\ \Theta_{\mathbb{Z}^{16}} \\ \Theta_{E_8 \oplus E_8} \\ \Theta_{D_{16}^+} \end{pmatrix} = \underline{0}.$$

As in the two previous cases, the same linear relations follow from the Table 1 of the Fourier coefficients of the lattice theta series considering the first five bold columns.

As for genus one and two we compute the Fourier coefficients for the functions $F_1^{(3)}$, $F_2^{(3)}$, $F_3^{(3)}$, $F_{16}^{(3)}$ and $F_{88}^{(3)}$ and we control the result using the computer. In Table 9 we show the result. We also compute the Fourier coefficients of the functions $F_8^{(3)}$ and $G_3^{(3)}[0^{(3)}]$. Thus, we get another proof of the relation (cf. [9], page 20):

$$G_3^{(3)}[0^{(3)}] = \frac{1}{64}F_8^{(3)} - \frac{1}{16}F_{88}^{(3)} - \frac{5}{448}(8F_{16}^{(3)} - F_8^{(3)}), \quad (5.4)$$

as can be check inserting in the previous equation the Fourier coefficients.

5.4 The case $g = 4$

The genus four case is the first interesting case because the factorization approach does no more work. The failure of this method is due to the fact that the space of moduli of curves is not the

Functions/m	(0, 0, 0)	(1, 0, 0)	(2, 0, 0)	(3, 0, 0)	(1, 1, 0)	(2, 2, 0)	(1, 1, 1)	(2, 2, 2)
F₁	1	32	480	4480	960	175680	26880	47174400
F₂	8	64	2304	33536	384	486912	1536	73451520
F₃	8	128	2816	34304	1792	727552	21504	137410560
F₁₆	8	0	3840	0	0	1405440	0	377395200
F₈	64	0	30720	0	0	11243520	0	3019161600
F₈₈	0	0	4096	-65536	0	2187264	0	673677312
G₃[0]	1	0	224	4096	0	38976	0	5069568

Table 9: Fourier coefficients for the F_1, F_2, F_3, F_{16} and F_{88} in genus three.

whole Siegel upper half plane. Indeed, the two theta series defined by the lattice D_{16}^+ and $E_8 \oplus E_8$ are no longer the same function and the differences among this two functions are lost by restricting on the boundary of \mathbb{H}_4 .

Thus, in order to find the relations between the lattice theta series and the functions $e_i^{(4)}$ we need the Fourier coefficients of the functions $e_i^{(4)}$. We have computed them with the C++ program. The result are reported in Table 10. Adding the rows of this table to the ones of Table 1 and

	(0, 0, 0, 0, 0)	(1, 0, 0, 0, 0)	(2, 0, 0, 0, 0)	(3, 0, 0, 0, 0)	(1, 1, 0, 0, 0)	(2, 2, 0, 0, 0)	(1, 1, 1, 0, 0)	(2, 2, 2, 0, 0)	(1, 1, 1, 1, 0)	(2, 2, 2, 2, 0)
F₁	1	32	480	4480	960	175680	26880	47174400	698880	8858304000
F₂	16	128	4608	67072	768	973824	3072	146903040	6144	15427215360
F₃	16	256	5632	68608	3584	1455104	43008	274821120	430080	37058273280
F₁₆	16	0	7680	0	0	2810880	0	754790400	0	141732864000
F₈	256	0	122880	0	0	44974080	0	12076646400	0	2320574054400
F₈₈	0	0	16384	-262144	0	8749056	0	2694709248	0	549726191616
G₃[0]	15	32	3616	61824	-64	655808	256	85511424	-1536	8099185152
G₄[0]	1	0	224	4096	0	38976	0	5069568	0	386797056
J⁽⁴⁾	0	0	0	0	0	0	0	0	0	-52848230400

Table 10: Fourier coefficients for the basis $F_1, F_2, F_3, F_{16}, F_{88}, F_8$ and $G_3[0]$ in genus four. In addition we compute the coefficients of $G_4[0]$ and of $J^{(4)}$.

considering the first seven bold columns, one finds, as expected, that the complete matrix has rank seven. Again, we get the expansions of the lattice theta series on the basis $e_i^{(4)}$. The result is shown in Table 11. These Fourier coefficients also provide a proof of the relation (cf. [9], page 23):

Theta series/Basis	F_1	F_2	F_3	F_{16}	F_{88}	F_8	$G_3[0]$
$\Theta_{(D_8 \oplus D_8)^+}$	0	0	0	0	-1/64	1/256	0
$\Theta_{\mathbb{Z} \oplus A_{15}^+}$	7/8192	35/4096	-45/32768	135/16384	-315/65536	45/65536	315/8192
$\Theta_{\mathbb{Z}^2 \oplus (E_7 \oplus E_7)^+}$	1/512	7/256	-7/2048	0	0	0	21/512
$\Theta_{\mathbb{Z}^4 \oplus D_{12}^+}$	0	1/16	0	0	0	0	0
$\Theta_{\mathbb{Z}^8 \oplus E_8}$	0	0	1/16	0	0	0	0
$\Theta_{\mathbb{Z}^{16}}$	1	0	0	0	0	0	0
$\Theta_{E_8 \oplus E_8}$	0	0	0	0	0	1/256	0
$\Theta_{D_{16}^+}$	0	0	0	1/16	0	0	0

Table 11: Theta series on the basis $F_1, F_2, F_3, F_{16}, F_{88}, F_8$ and $G_3[0]$ in genus four.

$$\begin{aligned}
G_4^{(4)}[0^{(4)}] &= \frac{1}{256}F_8^{(4)} - \frac{1}{64}F_{88}^{(4)} + \frac{3}{1792}J^{(4)} \\
&= \frac{1}{448}F_8^{(4)} - \frac{1}{64}F_{88}^{(4)} + \frac{3}{112}F_{16}^{(4)}.
\end{aligned}$$

Moreover, we obtain a linear relation between the lattice theta series

$$(1 \ -1024/315 \ 64/21 \ -8/9 \ 2/21 \ -1/315 \ -3/7 \ 3/7) \begin{pmatrix} \Theta_{(D_8 \oplus D_8)^+} \\ \Theta_{\mathbb{Z} \oplus A_{15}^+} \\ \Theta_{\mathbb{Z}^2 \oplus (E_7 \oplus E_7)^+} \\ \Theta_{\mathbb{Z}^4 \oplus D_{12}^+} \\ \Theta_{\mathbb{Z}^8 \oplus E_8} \\ \Theta_{\mathbb{Z}^{16}} \\ \Theta_{E_8 \oplus E_8} \\ \Theta_{D_{16}^+} \end{pmatrix} = \underline{0}.$$

5.5 The case $g = 5$

In genus five, we consider the eight columns of Table 1. This matrix has rank eight, so all the theta series are linearly independent. As in genus four, to study the relations between the Riemann theta constants and the lattice theta series we need the Fourier coefficients of the functions $e_i^{(5)}$. We have computed them by the computer and we report the result in Table 12, that also has rank eight. Gluing this table to the one of the Fourier coefficients for the lattice theta series we obtain a matrix

	(0, 0, 0, 0, 0)	(1, 0, 0, 0, 0)	(2, 0, 0, 0, 0)	(3, 0, 0, 0, 0)	(1, 1, 0, 0, 0)	(2, 2, 0, 0, 0)	(1, 1, 1, 0, 0)	(2, 2, 2, 0, 0)	(1, 1, 1, 1, 0)	(2, 2, 2, 2, 0)	(1, 1, 1, 1, 1)
F_1	1	32	480	4480	960	175680	26880	47174400	698880	8858304000	16773120
F_2	32	256	9216	134144	1536	1947648	6144	293806080	12288	30854430720	0
F_3	32	512	11264	137216	7168	2910208	86016	549642240	860160	74116546560	6881280
F_{16}	32	0	15360	0	0	5621760	0	1509580800	0	283465728000	0
F_8	1024	0	491520	0	0	179896320	0	48306585600	0	9282296217600	0
F_{88}	0	0	65536	-1048576	0	34996224	0	10778836992	0	2198904766464	0
$G_3[0]$	155	480	38560	640640	64	7174336	-2304	954147072	22016	90356353536	-225280
$G_4[0]$	31	32	7200	127360	-64	1279424	256	166624512	-1536	14287938048	12288
$J^{(5)}$	0	0	0	0	0	0	0	0	0	-211392921600	0

Table 12: Fourier coefficients for the $F_1, F_2, F_3, F_{16}, F_{88}, F_8, G_3[0]$ and $G_4[0]$ in genus five.

of rank eight. *So, all the lattice theta series can be expressed as linear combination of $e_i^{(5)}$ and vice versa!* Indeed we can be more precise. As the rank of the whole set of coefficients is 8, we get 8 linear relations among the two bases:

$$F_{16} = 2^5 \Theta_{D_{16}^+}, \quad F_8 = 2^{10} \Theta_{E_8 \oplus E_8}, \quad F_1 = \Theta_{\mathbb{Z}^{16}}, \quad (5.5)$$

$$F_3 = 2^5 \Theta_{\mathbb{Z}^8 \oplus E_8}, \quad F_2 = 2^5 \Theta_{\mathbb{Z}^4 \oplus D_{12}^+}, \quad F_8 - 4F_{88} = 2^{10} \Theta_{(D_8 \oplus D_8)^+}, \quad (5.6)$$

$$-4F_1 - 112F_2 + 7F_3 - 84G_3 = -16384 \Theta_{\mathbb{Z}^2 \oplus (E_7 \oplus E_7)^+}, \quad (5.7)$$

$$-28F_1 - 560F_2 + 45F_3 - 1260G_3 - 10080G_4 = -524288 \Theta_{\mathbb{Z} \oplus A_{15}^+}. \quad (5.8)$$

Note that the relations (5.5) can be directly checked. The relations (5.6) also are simply a generalization of the lower genus ones. However, for all the relations we can also give some consistency checks. Summing each side of the eight equalities over the 528 characteristics we obtain eight identities. For example for the (5.8) we obtain $-524288F_{16}^{(5)} = -524288 \cdot 2^5 \Theta_{D_{16}^+}$ and $2^5 \Theta_{D_{16}^+}$ is exactly the $F_{16}^{(5)}$. These sums can be performed using Table 6 of [9] and Table 1 and Appendix B.2 of [21]. Moreover, one verifies that also the restriction to $\mathbb{H}_1 \times \mathbb{H}_4$ of each equality is an identity.

6. Equivalence of the CDG and the OPSY construction

In this section we prove the equivalence of the two functions constructed using the classical theta functions and the lattice theta series. They at most differ by a multiple of the Schottky form and

become identical if one fixes the value of the cosmological constant to zero. We first study the Fourier coefficients of the two $\Xi_8^{(g)}[0^{(g)}]$, then we give an analytic proof of their equivalence.

6.1 Fourier coefficients for the partition function

Inserting the Fourier coefficients of the basis $e_i^{(g)}$ and of the lattice theta series in the definition of the functions $\Xi_8^{(g)}[0^{(g)}]$ of Section 4 we can compute, for every genus $g \leq 5$, the Fourier expansions of the $\Xi_8^{(g)}[0^{(g)}]$. Table 13 shows these coefficients for the two expressions of the forms Ξ_8 . We also add 0 in the first column for the functions $\Xi_8^{(g)}[0]_{OPSY}$, because, from the geometric discussion of Section 2, it is clear that there are no vectors in the lattice of null norm. We conclude that the

	(0, 0, 0, 0, 0)	(1, 0, 0, 0, 0)	(2, 0, 0, 0, 0)	(3, 0, 0, 0, 0)	(1, 1, 0, 0, 0)	(2, 2, 0, 0, 0)	(1, 1, 1, 0, 0)	(2, 2, 2, 0, 0)	(1, 1, 1, 1, 0)	(2, 2, 2, 2, 0)	(1, 1, 1, 1, 1)
$\Xi_8^{(1)}[0]_{OPSY}$	0	1	8	12							
$\Xi_8^{(1)}[0]_{CDG}$	0	16	128	192							
$\Xi_8^{(2)}[0]_{OPSY}$	0	0	0	0	1	64					
$\Xi_8^{(2)}[0]_{CDG}$	0	0	0	0	256	16384					
$\Xi_8^{(3)}[0]_{OPSY}$	0	0	0	0	0	0	1	192			
$\Xi_8^{(3)}[0]_{CDG}$	0	0	0	0	0	0	4096	786432			
$\Xi_8^{(4)}[0]_{OPSY}$	0	0	0	0	0	0	0	0	1	$\frac{38976}{17}$	
$\Xi_8^{(4)}[0]_{CDG}$	0	0	0	0	0	0	0	0	65536	$\frac{2554331136}{17}$	
$\Xi_8^{(5)}[0]_{OPSY}$	0	0	0	0	0	0	0	0	0	$\frac{16043183100}{11}$	1
$\Xi_8^{(5)}[0]_{CDG}$	0	0	0	0	0	0	0	0	0	$\frac{16822496762265600}{11}$	1048576

Table 13: Fourier coefficients for the two expressions of the form Ξ_8 . In the first line of each genus are the coefficients of the OPSY forms and in the second line the ones of the CDG forms.

two functions are the same up to genus five, apart for an unessential global factor 2^{4g} due to the different definition of the Dedekind function used in [9] and in [24] (cf. footnote 7 in [9], page 17).

6.2 Analytic proof of the equivalence of the CDG and the OPSY construction

In this section we give an analytic proof of the equivalence of the two constructions of the forms $\Xi_8[\Delta]$ through the study of their restriction to $\mathbb{H}_1 \times \mathbb{H}_4$. We will show that $(\Xi_8^{(5)}[0^{(5)}]_{CDG} - \Xi_8^{(5)}[0^{(5)}]_{OPSY})(\tau_{1,4}) = 0$ on the whole $\mathbb{H}_1 \times \mathbb{H}_4$. To compare the two expressions of the forms Ξ_8 one has to get rid of the factor 2^{4g} . We choose to multiply $\Xi_8^{(g)}[0]_{OPSY}$ by 2^{4g} and that implies that the constants b_4 and b_5 of Section 4 become $b_4 = -\frac{2^5 \cdot 3}{7 \cdot 17}$ and $b_5 = -\frac{2^5 \cdot 17}{7 \cdot 11}$. Indeed, using the expressions of Section 4:

$$\begin{aligned}
\Xi_8^{(5)}[0^{(5)}]_{OPSY}(\tau_{1,4}) &= \Xi_8^{(1)}[0^{(1)}](\tau_1) \Xi_8^{(4)}[0^{(4)}](\tau_4) \\
&\quad + \left(\frac{2^5 \cdot 3 \cdot 13}{7 \cdot 17} \Theta_{\mathbb{Z}^8 \oplus E_8}^{(1)} - \frac{2^6 \cdot 3^2 \cdot 5}{7 \cdot 17} \Theta_{\mathbb{Z}_{16}}^{(1)} + \frac{2^5 \cdot 17}{7 \cdot 11} \Theta_{E_8 \oplus E_8}^{(1)} \right) \left(\Theta_{E_8 \oplus E_8}^{(4)} - \Theta_{D_{16}^+}^{(4)} \right) \\
&= \Xi_8^{(1)}[0^{(1)}](\tau_1) \Xi_8^{(4)}[0^{(4)}](\tau_4) \\
&\quad + \left[\frac{3}{2^2 \cdot 7 \cdot 17} \left(-\frac{2}{3} F_1^{(1)} + 5 F_2^{(1)} \right) - \frac{17}{2^4 \cdot 7 \cdot 11} F_{16}^{(1)} \right] J^{(4)}, \tag{6.1}
\end{aligned}$$

where we have used the linear relation among the genus four lattice theta series found in 5.4, the genus one relations among the lattice theta series and the basis functions $e_i^{(4)}$ of Section 5.1, and the fact that⁵ $J^{(4)} = -2^8 (\Theta_{E_8 \oplus E_8}^{(4)} - \Theta_{D_{16}^+}^{(4)})$. With a similar computation we obtain for the form

⁵In general $J^{(g)} = -2^{2g} (\Theta_{E_8 \oplus E_8}^{(g)} - \Theta_{D_{16}^+}^{(g)})$.

$\Xi_8^{(5)}[0^{(5)}]_{CDG}$:

$$\begin{aligned}
\Xi_8^{(5)}[0^{(5)}]_{CDG}(\tau_{1,4}) &= \Xi_8^{(1)}[0^{(1)}](\tau_1)\Xi_8^{(4)}[0^{(4)}](\tau_4) \\
&+ \left(-\frac{3^2}{2^3 \cdot 7 \cdot 17} F_1^{(1)} + \frac{3 \cdot 5}{2^2 \cdot 7 \cdot 17} F_2^{(1)} - \frac{17}{2^4 \cdot 7 \cdot 11} F_{16}^{(1)} \right) J^{(4)} \\
&= \Xi_8^{(1)}[0^{(1)}](\tau_1)\Xi_8^{(4)}[0^{(4)}](\tau_4) \\
&+ \left[\frac{3}{2^2 \cdot 7 \cdot 17} \left(-\frac{2}{3} F_1^{(1)} + 5 F_2^{(1)} \right) - \frac{17}{2^4 \cdot 7 \cdot 11} F_{16}^{(1)} \right] J^{(4)}, \tag{6.2}
\end{aligned}$$

that is exactly the same as (6.1). This and the fact that the sum over the 528 genus five even characteristics of both the forms $\Xi_8^{(5)}[0^{(5)}]$ is a multiple of the Schottky form show the equivalence of the two constructions. Fixing the value of the cosmological constant and getting rid of the factor 2^{4g} , they do not differ neither for a multiple of $J^{(5)}$ because, if so, a term proportional to $F_{16}^{(1)} J^{(4)}$ should appear in the difference of their restrictions due to the fact that $J^{(5)}(\tau_{1,4}) = 2F_{16}^{(1)} J^{(4)}$. The factorizations can be obtained using the properties of the lattice theta series (see Section 2) and the restrictions properties of the functions $e_i^{(5)}$ (see [9], Section 4.2 and 7.1). Alternatively, one can employ the linear relations found in Section 5.5. Indeed changing the basis with those relations one obtains $\Xi_8^{(5)}[0^{(5)}]_{CDG}$ from $\Xi_8^{(5)}[0^{(5)}]_{OPSY}$ and vice versa. This is another check for the computation leading to relations (5.5), (5.6), (5.7) and (5.8).

7. Conclusions and perspectives

It is a well known fact (see [2, 3, 22, 23, 29]) that the path integral formulation of superstring theory at higher genus is affected by ambiguities, mainly due to the difficulty in finding a supercovariant formulation. Indeed, even though the super moduli space of super Riemann surfaces can be locally split in even and odd part, this does not work globally and the result comes out to depend on the choice of a bosonic slice in a non covariant way (see [5] for a review). In a series of papers [11–14], D’Hoker and Phong have been able to determine by direct calculation the genus two amplitudes. As a byproduct they formulated a set of ansätze that should be satisfied by the amplitudes at all genera. In [4] it has been shown that these ansätze characterize univocally the genus two measure, but they required a small modification to work at genus three [6, 10]. The same ansätze have produced solutions for the genus four and five cases also [7, 9].

The $g = 5$ case is particular, as resulted by the fact that fixing the Schottky term $J^{(5)}$ is necessary to get a vanishing cosmological constant. Note, however, that this should be a prediction of the theory and not an ansatz [20, 21]. This has been yet criticized in [17]. Thus, it becomes unclear if and what modular properties are sufficient to characterize the amplitudes. Such stronger constraints should come out from a more basis formulation of higher genus path integral. The ambiguity left open by the ansätze at $g \geq 5$ is an indetermination of the Schottky form contribution. This indetermination can be fixed by requiring also the vanishing of the cosmological constant. Note that the Schottky form in $g = 5$ does not vanish on the Jacobi locus [20], so the ambiguity is significative even on the locus of curves.

In [24] and [9] the solution of the constraints for $g = 5$ has been determined by means of different methods. In the first paper the authors started from a basis of the lattice theta series of weight eight $M_8^{\Theta_S}(\Gamma_5(2))$, whereas in the second paper the author starts from a basis of the genus five modular forms of weight eight $M_8^{\theta}(\Gamma_5(2))^{O^+}$. In each case it has been determined a unique solution modulo $J^{(5)}$. By computing the Fourier coefficients of both basis here we have been able to show that this two solutions coincide modulo $J^{(5)}$ and they become exactly the same if we impose the vanishing of the cosmological constant. Thus, we could suspect that a further constraint could imply uniqueness. Moreover, we have shown the complete equivalence for lower genus and we have

completely determined the relations between the corresponding selected basis, then determining an explicit identification of the spaces $M_8^{\Theta_S}(\Gamma_g(2))$ and $M_8^{\theta}(\Gamma_g(2))^{O^+}$ for $g = 1, \dots, 5$.

The previous considerations also lead to the question whether for $g > 5$ the ambiguity left open by the three constraints is again an indetermination of the Schottky form contribution or of stronger nature. Moreover, the trick of fixing the value of the cosmological constant does not work for $g > 5$, as pointed out in [17]. In addition, for $g \geq 5$, due to the non normality of the ring $M_8(\Gamma_g(2))$, might exist modular forms that are not polynomial in the theta constants satisfying the three constraints. The answer to this kind of questions would lead to a generalization of the uniqueness theorems proved up to genus four. Moreover, a deeper understanding of the path integral formulation of the theory is now essential to overcome these problems. However, it is worth to note that we are working here on the whole Siegel space, whereas string quantities require to be defined on the space of curves only (the Schottky locus). These points are actually under investigation.

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A. The program

In this section we briefly present the structure of the program we used to compute the Fourier coefficients of the functions $e_i^{(g)}$. The code is available on <http://www.dfm.uninsubria.it/thetac/>

An element of \mathbb{H}_g has the generic form:

$$\tau = \begin{pmatrix} \tau_1 & \tau_{g+1} & \cdots & \cdots & \tau_{2g-1} \\ \tau_{g+1} & \tau_2 & \tau_{2g} & \cdots & \tau_{3g-3} \\ \vdots & \vdots & \ddots & \dots & \vdots \\ \tau_{2g-2} & \tau_{3g-4} & \cdots & \tau_{g-1} & \tau_{g(g+1)/2} \\ \tau_{2g-1} & \tau_{3g-3} & \cdots & \cdots & \tau_g \end{pmatrix}. \quad (\text{A.1})$$

Thus, from the definition of theta constant (3.2) it is clear that truncating the series we obtain a polynomial in $g(g+1)/2$ variables $q_{ij} = e^{\pi i \tau_{ij}}$, with $1 \leq i \leq j \leq g$ and the same holds true for the functions $e_i^{(g)}$. It will be useful to rewrite the definition (3.2) as:

$$\theta_a^b(\tau) = (-)^{\frac{1}{2} \sum_i a_i b_i} \sum_{m \in \mathbb{Z}^g} (-)^{\sum_i m_i b_i} \left(\prod_i p_{ii}^{(2m_i + a_i)^2} \right) \left(\prod_{i < j} p_{ij}^{2(2m_i + a_i)(2m_j + a_j)} \right), \quad (\text{A.2})$$

with $p_{ij} = q_{ij}^{1/4}$, so the exponents are integer numbers. This renders faster the computations with the computer. The previous expansion may be thought as a polynomial in p_{ii} with coefficients that are polynomials in p_{ij} , $i < j$ (this observation will be useful later).

To perform the computation we have defined some C++ classes. First, we have defined the generic class `Polynomial`, defined as `template <typename CffType, typename ExpType> class Polynomial`, which accepts two types as parameters, `CffType` and `ExpType`. `CffType` represents the type of the coefficient of a single monomial in `Polynomial` and `ExpType` the type of the exponent. In order to perform the elementary operations with polynomials, we have introduced the operators of addition, multiplication and raising to power for the `Polynomial` class. Then, we have

defined a simple polynomial with integer coefficients: `typedef Polynomial<cln::cl_I, short> IntPol`⁶. This type will be the coefficient for the `ThetaPol` polynomial, which will be used to represent the series expansion of the theta constants: `typedef Polynomial<IntPol, unsigned short> ThetaPol`.

In order to compute the Fourier coefficients corresponding to the ten diagonal matrices of Section 2.2 we proceed as follows. For each even theta constant⁷ we “fill up” the `ThetaPol` by computing the (finite) sums (A.2) in which each component of $m \in \mathbb{Z}^g$ is no bigger than three. Using the operations on the polynomials we just defined, the `ThetaPol`’s are the bricks to build up the functions $e_i^{(g)}$ from their definition. Therefore, the function $e_i^{(g)}$ has the generic form:

$$e_i^{(g)}(\tau) = \sum_{n_1, \dots, n_g \in \mathbb{N}_0} (\dots) p_{11}^{n_1} \dots p_{gg}^{n_g}, \quad (\text{A.3})$$

where in (\dots) there are the non diagonal or constant terms. Note that the exponents of the diagonal terms p_{ii} are always positive, hence multiplying the polynomials of the theta constants the exponents cannot decrease. Due to our choice for the ten matrices, we can introduce a sort of “filter” for the value of the exponents. Roughly speaking, in the expansion (A.3) we neglect the terms with exponent of p_{ii} “bigger than the ones appearing in the diagonal of the ten matrices”. This allows us to make the computations very fast. Thus, the Fourier coefficients of the matrix $m = \text{diag}(m_1, \dots, m_g)$ is the constant term in (\dots) of the monomial with $n_1 = 4m_1, \dots, n_g = 4m_g$.

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⁶To manage long integer coefficients we use the `cl_I` class from CLN library, <http://www.ginac.de/CLN/>

⁷Recall that the number of even theta constants is $2^{g-1}(2^g+1)$.

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